Number fields with restricted ramification and rational points on stacks

Julie Tavernier

University of Bath

December 2024

Malle's conjecture

A conjecture by Malle predicts the number of number fields of fixed Galois group and bounded discriminant.

Conjecture (G. Malle, 2002)

G is a fixed finite group.

$$\#\left\{L/k : \begin{array}{c}[L:K]=n,\ |\Delta_{L/k}| \leq B \\ \mathrm{Gal}(L/k) \cong G\end{array}\right\} \sim c_{Malle}(k,G) B^{a(G)} (\log B)^{b(k,G)-1}$$

where $a(G) > 0$, $c_{Malle}(k,G) > 0$, $b(k,G) \in \mathbb{N}$.

Malle's conjecture

A conjecture by Malle predicts the number of number fields of fixed Galois group and bounded discriminant.

Conjecture (G. Malle, 2002)

G is a fixed finite group.

$$\#\left\{L/k : \begin{array}{c}[L:K]=n,\ |\Delta_{L/k}|\leq B \\ \mathrm{Gal}(L/k)\cong G\end{array}\right\} \sim c_{Malle}(k,G)B^{a(G)}(\log B)^{b(k,G)-1}$$

where $a(G)>0$, $c_{Malle}(k,G)>0$, $b(k,G)\in\mathbb{N}$.

Malle's prediction for $b(k,G)$ was found by Klüners to be incorrect and he did not put forward a prediction for the leading constant $c_{Malle}(k,G)$.

Malle's conjecture

A conjecture by Malle predicts the number of number fields of fixed Galois group and bounded discriminant.

Conjecture (G. Malle, 2002)

G is a fixed finite group.

$$\#\left\{L/k : \quad \begin{array}{c} [L:K]=n,\ |\Delta_{L/k}| \leq B \\ \mathrm{Gal}(L/k) \cong G \end{array} \right\} \sim c_{Malle}(k,G) B^{a(G)} (\log B)^{b(k,G)-1}$$

where $a(G) > 0$, $c_{Malle}(k,G) > 0$, $b(k,G) \in \mathbb{N}$.

Malle's prediction for $b(k,G)$ was found by Klüners to be incorrect and he did not put forward a prediction for the leading constant $c_{Malle}(k,G)$.

For the remainder of this talk G is a finite abelian group, k is a number field and Γ_k the absolute Galois group of k.

If v is a place of k, then k_v is the completion of k at v.

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant.

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

The ramification type is a homomorphism $\rho_G : \mathrm{Hom}(\Gamma_{k_v}, G) \to G(-1)$. We define $\rho_G(\varphi)$ as follows:

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \text{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

The ramification type is a homomorphism $\rho_G : \text{Hom}(\Gamma_{k_v}, G) \to G(-1)$. We define $\rho_G(\varphi)$ as follows:

- Since φ is continuous it factors through a finite tamely ramified extension L/k_v which we can assume contains μ_e.

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

The ramification type is a homomorphism $\rho_G : \mathrm{Hom}(\Gamma_{k_v}, G) \to G(-1)$. We define $\rho_G(\varphi)$ as follows:

- Since φ is continuous it factors through a finite tamely ramified extension L/k_v which we can assume contains μ_e.
- There is an isomorphism $\mu_e \to I_v$ where I_v is the inertia group of $\mathrm{Gal}(L/k_v)$.

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

The ramification type is a homomorphism $\rho_G : \mathrm{Hom}(\Gamma_{k_v}, G) \to G(-1)$. We define $\rho_G(\varphi)$ as follows:

- Since φ is continuous it factors through a finite tamely ramified extension L/k_v which we can assume contains μ_e.
- There is an isomorphism $\mu_e \to I_v$ where I_v is the inertia group of $\mathrm{Gal}(L/k_v)$.
- We can compose this with φ to obtain a homomorphism $\mu_e \to G$ which is an element of $G(-1)$.

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

The ramification type is a homomorphism $\rho_G : \mathrm{Hom}(\Gamma_{k_v}, G) \to G(-1)$. We define $\rho_G(\varphi)$ as follows:

- Since φ is continuous it factors through a finite tamely ramified extension L/k_v which we can assume contains μ_e.
- There is an isomorphism $\mu_e \to I_v$ where I_v is the inertia group of $\mathrm{Gal}(L/k_v)$.
- We can compose this with φ to obtain a homomorphism $\mu_e \to G$ which is an element of $G(-1)$.

This is $\rho_G(\varphi)$.

Ramification type

Malle's conjecture counts number fields by discriminant. However it is possible to count using more general height functions than just the discriminant. Let $G(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), G)$ where $\hat{\mathbb{Z}}(1) = \lim_{\leftarrow n} \mu_n$.

Ramification type

The ramification type is a homomorphism $\rho_G : \mathrm{Hom}(\Gamma_{k_v}, G) \to G(-1)$. We define $\rho_G(\varphi)$ as follows:

- Since φ is continuous it factors through a finite tamely ramified extension L/k_v which we can assume contains μ_e.
- There is an isomorphism $\mu_e \to I_v$ where I_v is the inertia group of $\mathrm{Gal}(L/k_v)$.
- We can compose this with φ to obtain a homomorphism $\mu_e \to G$ which is an element of $G(-1)$.

This is $\rho_G(\varphi)$.

Then the height of φ is $H(\varphi) = \prod_v H_v(\varphi)$ where each H_v is

$$H_v(\varphi) = q_v^{w(\rho_G(\varphi))}$$

where $w : G(-1) \to \mathbb{Q}$ is a class function and q_v is the size of the residue field.

Counting with local conditions

Rather than count all number fields as in Malle's conjecture, one could restrict their attention to only counting number fields with some sort of imposed condition such as restricted ramification.

Counting with local conditions

Rather than count all number fields as in Malle's conjecture, one could restrict their attention to only counting number fields with some sort of imposed condition such as restricted ramification.

A finite abelian number field with Galois group G corresponds to a surjective homomorphism $\Gamma_k \to G$, called a G-extension of k. A sub-G-extension of k is a continuous homomorphism $\Gamma_k \to G$.

Counting with local conditions

Rather than count all number fields as in Malle's conjecture, one could restrict their attention to only counting number fields with some sort of imposed condition such as restricted ramification.

A finite abelian number field with Galois group G corresponds to a surjective homomorphism $\Gamma_k \to G$, called a G-extension of k. A sub-G-extension of k is a continuous homomorphism $\Gamma_k \to G$.

Let H be a height function with corresponding weight function w and take a non-trivial set $R \subseteq G(-1)$. Consider the subset $M_R(H) = \{\gamma \in R : w(\gamma) \text{ is minimal}\}$. Let S be a finite set containing "bad" places. Define $W = (W_v)_{v \in \Omega_k}$ as follows:

Counting with local conditions

Rather than count all number fields as in Malle's conjecture, one could restrict their attention to only counting number fields with some sort of imposed condition such as restricted ramification.

A finite abelian number field with Galois group G corresponds to a surjective homomorphism $\Gamma_k \to G$, called a G-extension of k. A sub-G-extension of k is a continuous homomorphism $\Gamma_k \to G$.

Let H be a height function with corresponding weight function w and take a non-trivial set $R \subseteq G(-1)$. Consider the subset $M_R(H) = \{\gamma \in R : w(\gamma) \text{ is minimal}\}$. Let S be a finite set containing "bad" places. Define $W = (W_v)_{v \in \Omega_k}$ as follows:

- For $v \in S$ let W_v be a non-empty set of sub-G-extensions of k_v.

Counting with local conditions

Rather than count all number fields as in Malle's conjecture, one could restrict their attention to only counting number fields with some sort of imposed condition such as restricted ramification.

A finite abelian number field with Galois group G corresponds to a surjective homomorphism $\Gamma_k \to G$, called a G-extension of k. A sub-G-extension of k is a continuous homomorphism $\Gamma_k \to G$.

Let H be a height function with corresponding weight function w and take a non-trivial set $R \subseteq G(-1)$. Consider the subset $M_R(H) = \{\gamma \in R : w(\gamma) \text{ is minimal}\}$. Let S be a finite set containing "bad" places. Define $W = (W_v)_{v \in \Omega_k}$ as follows:

- For $v \in S$ let W_v be a non-empty set of sub-G-extensions of k_v.
- For $v \notin S$ let W_v be the set of sub-G-extensions of k_v such that $\rho_G(\varphi_v) \in M_R(H) \cup \{1\}$.

Counting with restricted ramification

We count these G-extensions (corresponding to number fields) with ramification imposed by the set R.

Counting with restricted ramification

We count these G-extensions (corresponding to number fields) with ramification imposed by the set R.

Theorem (T. 2024)

Counting with restricted ramification

We count these G-extensions (corresponding to number fields) with ramification imposed by the set R.

Theorem (T. 2024)

We have

$$N(k, R, H, B) = \#\{\varphi \in G\text{-ext}(k) : H(\varphi) \leq B, \varphi_v \in W_v \, \forall v\} \sim c(k, R, H) B^{a_R(H)} (\log B)^{b_R(H)-1}$$

Counting with restricted ramification

We count these G-extensions (corresponding to number fields) with ramification imposed by the set R.

Theorem (T. 2024)

We have

$$N(k, R, H, B) = \#\{\varphi \in G\text{-ext}(k) : H(\varphi) \leq B, \varphi_v \in W_v \; \forall v\} \sim c(k, R, H) B^{a_R(H)} (\log B)^{b_R(H)-1}$$

where $a_R(H) = (\min_{1 \neq g \in R} w(g))^{-1}$ *and* $b_R(H) = \#M_R(H)/\Gamma_k$.

Counting with restricted ramification

We count these G-extensions (corresponding to number fields) with ramification imposed by the set R.

Theorem (T. 2024)

We have

$$N(k,R,H,B) = \#\{\varphi \in G\text{-ext}(k) : H(\varphi) \le B, \varphi_v \in W_v \,\forall v\} \sim c(k,R,H) B^{a_R(H)} (\log B)^{b_R(H)-1}$$

where $a_R(H) = (\min_{1\neq g\in R} w(g))^{-1}$ *and* $b_R(H) = \#M_R(H)/\Gamma_k$. *The leading constant is given by*

$$c(k,R,H) = \frac{a_R(H)^{b_R(H)-1}(\operatorname{Res}_{s=1}\zeta_k(s))^{b_R(H)}}{\Gamma(b_R(H))|\mathcal{O}_k^\times \otimes G^\wedge|} \times \sum_{x\in\mathcal{X}(k,R,H)} \prod_v \frac{1}{|G|} \sum_{\varphi_v\in W} \frac{\langle \varphi_v, x_v\rangle}{H_v(\varphi_v)^{a_R(H)}} \zeta_{k,v}(1)^{-b_R(H)}. \quad (1.1)$$

Counting with restricted ramification

What this means

This technical theorem just tells us that the number of G-extensions with ramification conditions imposed by the set R grows like $cB^{a_R(H)}(\log B)^{b_R(H)-1}$ for a positive constant c.

Counting with restricted ramification

What this means

This technical theorem just tells us that the number of G-extensions with ramification conditions imposed by the set R grows like $cB^{a_R(H)}(\log B)^{b_R(H)-1}$ for a positive constant c.

This can be proved using techniques taken from harmonic analysis.

Natural questions

- How does this count of G-extensions with restricted ramification compare to the total count of G-extensions of bounded height?

Natural questions

- How does this count of G-extensions with restricted ramification compare to the total count of G-extensions of bounded height?
- Does the quotient of the number of G-extensions with restricted ramification by the total number of G-extensions of bounded height have the expected local behaviour?

Natural questions

- How does this count of G-extensions with restricted ramification compare to the total count of G-extensions of bounded height?
- Does the quotient of the number of G-extensions with restricted ramification by the total number of G-extensions of bounded height have the expected local behaviour?

These sorts of questions of imposing "local specifications" and comparing to the total count of number fields is (informally) referred to as the Malle-Bhargava heuristics.

The stack BG

BG is the quotient stack that classifies G-torsors and their automorphisms.

The stack BG

BG is the quotient stack that classifies G-torsors and their automorphisms.

We are interested in the k-rational points of BG.

The stack BG

BG is the quotient stack that classifies G-torsors and their automorphisms.

We are interested in the k-rational points of BG.

If k is a number field and G a finite abelian group, the groupoid $BG(k)$ corresponds to the groupoid of homomorphisms $\mathrm{Hom}(\Gamma_k, G)$ with isomorphisms given by conjugation in G (trivial for a finite abelian group G).

The stack BG

BG is the quotient stack that classifies G-torsors and their automorphisms.

We are interested in the k-rational points of BG.

If k is a number field and G a finite abelian group, the groupoid $BG(k)$ corresponds to the groupoid of homomorphisms $\mathrm{Hom}(\Gamma_k, G)$ with isomorphisms given by conjugation in G (trivial for a finite abelian group G).

Fact - due to Loughran and Santens

Counting G-extensions for a finite abelian group G corresponds to counting rational points on the stack BG.

So what about counting rational points?

When it comes to counting rational points we have the following important open conjecture.

Conjecture (Y. Manin, Y. Tschinkel, J. Franke, V. Batyrev)

Let k be a number field, X a Fano variety and H a height function. There is a thin subset Z of rational points such that for $U = X \backslash Z$ we have

$$\#\{x \in U(k) : H(x) \leq B\} \sim c_{Manin} B^{a(D)} (\log B)^{b(k,D)-1}.$$

So what about counting rational points?

When it comes to counting rational points we have the following important open conjecture.

Conjecture (Y. Manin, Y. Tschinkel, J. Franke, V. Batyrev)

Let k be a number field, X a Fano variety and H a height function. There is a thin subset Z of rational points such that for $U = X \backslash Z$ we have

$$\#\{x \in U(k) : H(x) \leq B\} \sim c_{Manin} B^{a(D)} (\log B)^{b(k,D)-1}.$$

The asymptotic formula resembles that of Malle's conjecture.

Ellenberg, Satriano and Zureick-Brown, and later Darda and Yasuda have developed a version of the Batyrev-Manin conjecture on stacks.

So what about counting rational points?

When it comes to counting rational points we have the following important open conjecture.

Conjecture (Y. Manin, Y. Tschinkel, J. Franke, V. Batyrev)

Let k be a number field, X a Fano variety and H a height function. There is a thin subset Z of rational points such that for $U = X \backslash Z$ we have

$$\#\{x \in U(k) : H(x) \leq B\} \sim c_{Manin} B^{a(D)} (\log B)^{b(k,D)-1}.$$

The asymptotic formula resembles that of Malle's conjecture.

Ellenberg, Satriano and Zureick-Brown, and later Darda and Yasuda have developed a version of the Batyrev-Manin conjecture on stacks.

So...

So Malle's conjecture for G-extensions corresponds to Manin's conjecture on BG for a finite abelian group G?

Rephrasing our problem in Manin's conjecture language

A sub-G-extension φ_v of k_v corresponds to an element of $BG(k_v)$.

Rephrasing our problem in Manin's conjecture language

A sub-G-extension φ_v of k_v corresponds to an element of $BG(k_v)$. Let S be the same set of "bad" places. Then for all $v \notin S$ let

$$BG(\mathcal{O}_v)_{M_R(H)} = \{\varphi_v \in BG(k_v) : \rho_G(\varphi_v) \in M_R(H) \cup \{1\}\}.$$

Rephrasing our problem in Manin's conjecture language

A sub-G-extension φ_v of k_v corresponds to an element of $BG(k_v)$. Let S be the same set of "bad" places. Then for all $v \notin S$ let

$$BG(\mathcal{O}_v)_{M_R(H)} = \{\varphi_v \in BG(k_v) : \rho_G(\varphi_v) \in M_R(H) \cup \{1\}\}.$$

The set W from previously can be written as

$$W = \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}.$$

Then the counting function $N(k, R, H, B)$ can be written as

$$N(k, R, H, B) = \#\{\varphi \in BG[k] \backslash \Omega : \varphi \in W, H(\varphi) \leq B\}.$$

Rephrasing our problem in Manin's conjecture language

A sub-G-extension φ_v of k_v corresponds to an element of $BG(k_v)$. Let S be the same set of "bad" places. Then for all $v \notin S$ let

$$BG(\mathcal{O}_v)_{M_R(H)} = \{\varphi_v \in BG(k_v) : \rho_G(\varphi_v) \in M_R(H) \cup \{1\}\}.$$

The set W from previously can be written as

$$W = \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}.$$

Then the counting function $N(k, R, H, B)$ can be written as

$$N(k, R, H, B) = \#\{\varphi \in BG[k] \backslash \Omega : \varphi \in W, H(\varphi) \leq B\}.$$

What is Ω?

We only want to count G-extensions (*surjective* continuous homomorphisms). In Manin's conjecture literature we must remove a thin set of "bad" rational points. Ω corresponds to this thin set and contains the non-surjective homomorphisms.

Equidistribution

We mentioned the "Malle-Bhargava heuristics" earlier. In the language of Manin's conjecture, this is referred to as "equidistribution". Studying the equidistribution of rational points means to study this limit

$$\lim_{B\to\infty} \frac{\#\{\varphi \in BG[k]\backslash\Omega : \varphi \in W, H(\varphi) \leq B\}}{\#\{\varphi \in BG[k]\backslash\Omega : H(\varphi) \leq B\}}.$$

To do so we need to understand the leading constant better.

The leading constant again

Leading constant

$$c(k,R,H) = \frac{a_R(H)^{b_R(H)-1}(\mathrm{Res}_{s=1}\,\zeta_k(s))^{b_R(H)}}{\Gamma(b_R(H))|\mathcal{O}_k^\times \otimes G^\wedge|} \sum_{x\in\mathcal{X}(k,R,H)} \prod_v \frac{1}{|G|} \sum_{\varphi_v \in W_v} \frac{\langle \varphi_v, x_v\rangle}{H_v(\varphi_v)^{a_R(H)}\zeta_{k,v}(1)^{b_R(H)}}$$

where

$$\mathcal{X}(k,R,H) = \{x \in k^\times \otimes G^\wedge : \text{for all but finitely many } v \text{ such that } \rho_G(\varphi_v) \in M_R(H)\cup\{e\}, \text{ we have } x_v = 1 \in k_v^\times \otimes G^\wedge\}.$$

Questions one might ask regarding the leading constant:

The leading constant again

Leading constant

$$c(k,R,H) = \frac{a_R(H)^{b_R(H)-1}(\mathrm{Res}_{s=1}\,\zeta_k(s))^{b_R(H)}}{\Gamma(b_R(H))|\mathcal{O}_k^\times \otimes G^\wedge|} \sum_{x\in\mathcal{X}(k,R,H)} \prod_v \frac{1}{|G|} \sum_{\varphi_v \in W_v} \frac{\langle \varphi_v, x_v\rangle}{H_v(\varphi_v)^{a_R(H)}\zeta_{k,v}(1)^{b_R(H)}}$$

where

$$\mathcal{X}(k,R,H) = \{x \in k^\times \otimes G^\wedge : \text{for all but finitely many } v \text{ such that } \rho_G(\varphi_v) \in M_R(H) \cup \{e\}, \text{ we have } x_v = 1 \in k_v^\times \otimes G^\wedge\}.$$

Questions one might ask regarding the leading constant:

- Is it always positive?

The leading constant again

Leading constant

$$c(k,R,H) = \frac{a_R(H)^{b_R(H)-1}(\mathrm{Res}_{s=1}\,\zeta_k(s))^{b_R(H)}}{\Gamma(b_R(H))|\mathcal{O}_k^\times \otimes G^\wedge|} \sum_{x\in\mathcal{X}(k,R,H)} \prod_v \frac{1}{|G|} \sum_{\varphi_v \in W_v} \frac{\langle \varphi_v, x_v\rangle}{H_v(\varphi_v)^{a_R(H)}\zeta_{k,v}(1)^{b_R(H)}}$$

where

$$\mathcal{X}(k,R,H) = \{x \in k^\times \otimes G^\wedge : \text{for all but finitely many } v \text{ such that } \rho_G(\varphi_v) \in M_R(H) \cup \{e\}, \text{ we have } x_v = 1 \in k_v^\times \otimes G^\wedge\}.$$

Questions one might ask regarding the leading constant:

- Is it always positive?
- If not, what conditions are necessary for it to be positive?

The leading constant again

Leading constant

$$c(k,R,H) = \frac{a_R(H)^{b_R(H)-1}(\mathrm{Res}_{s=1}\,\zeta_k(s))^{b_R(H)}}{\Gamma(b_R(H))|\mathcal{O}_k^\times \otimes G^\wedge|} \sum_{x\in\mathcal{X}(k,R,H)} \prod_v \frac{1}{|G|} \sum_{\varphi_v\in W_v} \frac{\langle \varphi_v, x_v\rangle}{H_v(\varphi_v)^{a_R(H)}\zeta_{k,v}(1)^{b_R(H)}}$$

where

$$\mathcal{X}(k,R,H) = \{x \in k^\times \otimes G^\wedge : \text{for all but finitely many } v \text{ such that } \rho_G(\varphi_v) \in M_R(H)\cup\{e\}, \text{ we have } x_v = 1 \in k_v^\times \otimes G^\wedge\}.$$

Questions one might ask regarding the leading constant:

- Is it always positive?
- If not, what conditions are necessary for it to be positive?

With a bit of thinking, one might see that the positivity of the leading constant is controlled by the Pontryagin pairing $\langle \varphi_v, x_v\rangle$, where $\varphi_v \in W_v$ and $x \in \mathcal{X}(k,R,H)$.

The partially ramified Brauer group

Suppose $b \in \mathrm{Br}\, BG$. We say that b is in the partially ramified Brauer group $\mathrm{Br}_{M_R(H)}\, BG$ if it evaluates trivially on $BG(\mathcal{O}_v)_{M_R(H)}$ for all but finitely many v, where

$$BG(\mathcal{O}_v)_{M_R(H)} = \{\varphi_v \in BG(k_v) : \rho_G(\varphi_v) \in M_R(H) \cup \{1\}\}.$$

The partially ramified Brauer group

Suppose $b \in \operatorname{Br} BG$. We say that b is in the partially ramified Brauer group $\operatorname{Br}_{M_R(H)} BG$ if it evaluates trivially on $BG(\mathcal{O}_v)_{M_R(H)}$ for all but finitely many v, where

$$BG(\mathcal{O}_v)_{M_R(H)} = \{\varphi_v \in BG(k_v) : \rho_G(\varphi_v) \in M_R(H) \cup \{1\}\}.$$

Lemma

The set $\mathcal{X}(k, R, H)$ is identified with the partially ramified Brauer group $\operatorname{Br}_{M_R(H)} BG / \operatorname{Br} k$ and we have an identification between

$$\sum_{x \in \mathcal{X}(k,R,H)} \langle \varphi_v, x_v \rangle \quad \text{and} \quad \sum_{b \in \operatorname{Br}_{M_R(H)} BG / \operatorname{Br} k} e^{2\pi i \langle b, \varphi \rangle_{BM}},$$

where $\langle -, - \rangle_{BM}$ is the Brauer-Manin pairing on BG.

Positivity of the leading constant

Consider

$$\sum_{b \in \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k} e^{2\pi i \langle b, \varphi \rangle_{BM}}$$

where $\varphi \in BG(\mathbb{A}_k)_{M_R(H)} = \lim_S \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}$.

Positivity of the leading constant

Consider

$$\sum_{b \in \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k} e^{2\pi i \langle b, \varphi \rangle_{BM}}$$

where $\varphi \in BG(\mathbb{A}_k)_{M_R(H)} = \lim_S \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}$.

- This is non-zero when $\varphi \in BG(\mathbb{A}_k)^{\mathrm{Br}}_{M_R(H)}$ where $\mathrm{Br} = \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k$, this is the Brauer-Manin set of $BG(\mathbb{A}_k)_{M_R(H)}$.

Positivity of the leading constant

Consider

$$\sum_{b \in \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k} e^{2\pi i \langle b, \varphi \rangle_{BM}}$$

where $\varphi \in BG(\mathbb{A}_k)_{M_R(H)} = \lim_S \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}$.

- This is non-zero when $\varphi \in BG(\mathbb{A}_k)^{\mathrm{Br}}_{M_R(H)}$ where $\mathrm{Br} = \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k$, this is the Brauer-Manin set of $BG(\mathbb{A}_k)_{M_R(H)}$.
- This is identified with $\sum_{x \in \mathcal{X}(k,R,H)} \langle \chi_v, x_v \rangle$ appearing in the leading constant.

Positivity of the leading constant

Consider

$$\sum_{b \in \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k} e^{2\pi i \langle b, \varphi \rangle_{BM}}$$

where $\varphi \in BG(\mathbb{A}_k)_{M_R(H)} = \lim_S \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}$.

- This is non-zero when $\varphi \in BG(\mathbb{A}_k)^{\mathrm{Br}}_{M_R(H)}$ where $\mathrm{Br} = \mathrm{Br}_{M_R(H)} BG / \mathrm{Br}\, k$, this is the Brauer-Manin set of $BG(\mathbb{A}_k)_{M_R(H)}$.
- This is identified with $\sum_{x \in \mathcal{X}(k,R,H)} \langle \chi_v, x_v \rangle$ appearing in the leading constant.

Upshot

The positivity of the leading constant is controlled by a Brauer-Manin obstruction on BG!

Positivity of the leading constant

Consider

$$\sum_{b \in \operatorname{Br}_{M_R(H)} BG/\operatorname{Br} k} e^{2\pi i \langle b, \varphi \rangle_{BM}}$$

where $\varphi \in BG(\mathbb{A}_k)_{M_R(H)} = \lim_S \prod_{v \in S} BG(k_v) \prod_{v \notin S} BG(\mathcal{O}_v)_{M_R(H)}$.

- This is non-zero when $\varphi \in BG(\mathbb{A}_k)^{\operatorname{Br}}_{M_R(H)}$ where $\operatorname{Br} = \operatorname{Br}_{M_R(H)} BG/\operatorname{Br} k$, this is the Brauer-Manin set of $BG(\mathbb{A}_k)_{M_R(H)}$.
- This is identified with $\sum_{x \in \mathcal{X}(k,R,H)} \langle \chi_v, x_v \rangle$ appearing in the leading constant.

Upshot

The positivity of the leading constant is controlled by a Brauer-Manin obstruction on BG!

In particular, one can show that the leading constant can be written as

$$c(k,R,H) = \frac{a_R(H)^{b_R(H)-1} \tau_H(W \cap BG(\mathbb{A}_k)^{\operatorname{Br}}_{M_R(H)}) |\operatorname{Br}_{M_R(H)} BG/\operatorname{Br} k|}{|G^\wedge(k)| \Gamma(b_R(H))},$$

where τ_H is a Tamagawa measure defined on BG.

Equidistribution

Finally, recall a continuity set is a set whose boundary has measure 0.

Equidistribution

Finally, recall a continuity set is a set whose boundary has measure 0.

Theorem (T. 2024)

Let R be a non-trivial subset of $G(-1)$ such that $M_R(H)$ generates $G(-1)$ and $W_R \subset BG(\mathbb{A}_k)_{M_R(H)}$ be a continuity set. Then there exists a thin subset $\Omega \subset BG[k]$ such that

$$\lim_{B\to\infty} \frac{\#\{\varphi \in BG[k]\backslash\Omega : \varphi \in W_R, H(\varphi) \leq B\}}{\#\{\varphi \in BG[k]\backslash\Omega : H(\varphi) \leq B\}} = \frac{\tau_H(W_R \cap BG(\mathbb{A}_k)^{\mathrm{Br}}_{M_R(H)})}{\tau_H(BG(\mathbb{A}_k)^{\mathrm{Br}}_{M_R(H)})}.$$

This is proved using the formula in terms of Brauer groups and Tamagawa measures for the leading constant.